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# Free Variables and the Two Matrix Model

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We study the full set of planar Green's functions for a two-matrix model using the language of functions of non-commuting variables. Both the standard Schwinger-Dyson equations and equations determining connected Green's functions can be efficiently discussed and solved. This solution determines the master field for the model in the ' $C$ -representation.'

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A recent convergence of work by mathematicians [1,2] and physicists [3,4,5,6,7] has produced precise constructions of the ‘master field’ for general large  $N$  gauge and matrix field theories. The essential difficulty of the large number of degrees of freedom in higher dimensional large  $N$  theories is dealt with by finding master fields which live in ‘large’ operator algebras such as the type  $\text{II}_1$  factor associated with a free group.

When we say that the master field was ‘constructed’ in a higher dimensional theory, so far what we mean is that if we know all correlation functions for the theory, we can produce an operator representation of the master field. Thus to begin we should study models we can already solve, and see if any simplifications appear in the new language. The large number of degrees of freedom of higher dimensions (and the need to use non-trivial free algebras) appears in the large  $N$  limit of any integral over more than one matrix, and the simplest example is the two-matrix model defined by the large  $N$  limit of the integral

$$Z = \int \mathcal{D}M_1 \mathcal{D}M_2 e^{-N\text{Tr} [V(M_1)+V(M_2)-cM_1M_2]} \quad (1)$$

with  $M_i$  hermitian. We will take  $V$  to be the cubic polynomial  $V(M) = \frac{1}{2}M^2 - \frac{g}{3}M^3$  below, but our considerations generalize easily to higher order polynomials. This model was first solved in [9] using orthogonal polynomials. This technique uses in an essential way the special form of the action in (1) and the ability to express the solution in terms of the invariant observables constructed from a single matrix. This generalizes to higher dimensions only in very special cases such as the Kazakov-Migdal model [12] and we would like to explore techniques which do not depend so fundamentally on a particular form for the action.

One such technique is to solve the factorized Schwinger-Dyson equations. At present, one can do this explicitly only when one can find a subset of these equations which closes on a few invariants, and such a reduction for (1) was found in [10,11]. We begin by discussing this and the recursive definition of higher correlation functions in the language of [4]. The generating function of planar Green’s functions is a function of non-commuting variables

$$\phi(u_1, u_2) = \sum_{\text{words } w} w(u_1, u_2) \langle \frac{1}{N} \text{Tr } w(M_1, M_2) \rangle. \quad (2)$$

The Green’s functions and thus  $\phi$  have cyclic symmetry.

The Schwinger-Dyson equations for (1) then read

$$\begin{aligned} \partial_1 \phi - c \partial_2 \phi - g \partial_1^2 \phi &= \phi u_1 \phi \\ \partial_2 \phi - c \partial_1 \phi - g \partial_2^2 \phi &= \phi u_2 \phi. \end{aligned} \quad (3)$$

Derivatives acting on functions of free variables satisfy

$$\frac{\partial}{\partial u_i} u_j f(u) = \partial_i u_j f(u) = \delta_{ij} f(u) \quad (4)$$

and this is just an alternate notation  $\partial_i = a_i$ ,  $u_i = a_i^*$  for the free algebra of [1]. We also define right derivatives

$$f(u) u_j \overleftarrow{\frac{\partial}{\partial u_i}} = f(u) u_j \overleftarrow{\partial}_i = \delta_{ij} f(u). \quad (5)$$

Acting on a function with cyclic symmetry (such as  $\phi$ ) these satisfy

$$\partial \dots f(u) \overleftarrow{\partial}_i = \partial \dots \partial_i f(u). \quad (6)$$

To get a truncated system of equations we work with the first terms in the series expansion

$$\sum_{k \geq 0} u_2^k \partial_2^k \phi(u_1, u_2) \big|_{u_2=0} = \phi_0(u_1) + u_2 \phi_1(u_1) + u_2^2 \phi_2(u_1) + \dots, \quad (7)$$

consider the equations (3) and their derivatives at  $u_2 = 0$ , and look for a minimal set of closed equations. Identifying terms of the form (7) in (3) we get

$$\partial_1 \phi_k - c \phi_{k+1} - g \partial_1^2 \phi_k = \phi_k u_1 \phi_0 \quad (8)$$

$$\phi_1 - c \partial_1 \phi_0 - g \phi_2 = 0. \quad (9)$$

From (9) a minimal closed set will be  $\{\phi_0, \phi_1, \phi_2\}$  and including (8) with  $k = 0$  and  $k = 1$  gives three equations. The operator  $\partial_1$  is then rewritten

$$\partial_1^n f(u_1) = \left[ \frac{1}{u_1^n} f(u_1) \right]_+ = \frac{1}{u_1^n} \left( f(u_1) - \sum_{i=0}^{n-1} u_1^i \partial_1^i f(0) \right) \quad (10)$$

producing algebraic equations with boundary conditions  $\partial_1^i \phi(0)$  for  $i \leq 3$ . These can be combined into a cubic equation for  $\phi_0(u_1)$ , to be made more explicit below. In general, we get an equation of degree  $\deg V$ .

This is just a start on the problem of getting all correlation functions and the simplest description of these would be a single equation for  $\phi$  which ‘closes.’ Not having very sophisticated techniques for working with functions of non-commuting variables, what we will mean by this is that we can directly truncate this single equation to an equation for  $\phi_0(u_1)$  and then solve iteratively for  $\phi$  in an expansion with terms containing  $u_2$ ’s in all possible ways.

We can find such an equation by repeating the strategy of taking derivatives of (3) and eliminating common factors in the result, but this time not setting  $u_2 = 0$ . A useful set of equations derived from (3) is

$$\partial_2 \partial_1 \phi - g \partial_2 \partial_1^2 \phi - \frac{c}{g} (\partial_2 \phi - c \partial_1 \phi - \phi u_2 \phi) = \partial_2 \phi u_1 \phi, \quad (11)$$

and equations derived by acting with the right derivative  $\overleftarrow{\partial}_1$

$$\begin{aligned} \partial_2 \phi &= \frac{1}{c} (\partial_1 \phi - g \partial_1^2 \phi - \phi u_1 \phi) \\ \partial_2 \partial_1 \phi &= \frac{1}{c} (\partial_1^2 \phi - g \partial_1^3 \phi - \phi u_1 \partial_1 \phi - \phi) \\ \partial_2 \partial_1^2 \phi &= \frac{1}{c} (\partial_1^3 \phi - g \partial_1^4 \phi - \phi u_1 \partial_1^2 \phi - \partial_1 \phi(0) \phi - \partial_1 \phi). \end{aligned} \quad (12)$$

The final equation is obtained by substituting eqs.(12) into (11):

$$D\phi = -\frac{c^2}{g} \phi u_2 \phi - \frac{c}{g} \phi u_1 \phi + \partial_1 \phi u_1 \phi + \phi u_1 \partial_1 \phi - g \partial_1^2 \phi u_1 \phi - g \phi u_1 \partial_1^2 \phi - \phi u_1 \phi u_1 \phi. \quad (13)$$

where the operator  $D$  is given by

$$D = -1 + g \partial_1 \phi(0) + (g - \frac{c}{g} + \frac{c^3}{g}) \partial_1 + (1 + c) \partial_1^2 - 2g \partial_1^3 + g^2 \partial_1^4.$$

First let  $u_2 = 0$  in (13), to obtain a cubic equation for  $\phi_0(u_1)$ . Let  $R(z) = 1/z \phi_0(1/z)$ , this algebraic equation is re-written as

$$\begin{aligned} F(z, R) &\equiv R^3 - f(z)R^2 + g(z)R - h(z) = 0, \\ f(z) &= -\frac{c}{g} + 2z - 2gz^2, \\ g(z) &= g^2 z^4 - 2gz^3 + (1 + c)z^2 - (g + c/g - c^3/g^2)z + 1 - g \partial_1 \phi(0), \\ h(z) &= g^2 z^3 + (g^2 \partial_1 \phi(0) - 2g)z^2 + (g^2 \partial_1^2 \phi(0) - 2g \partial_1 \phi(0) - 1 - c)z \\ &\quad + g^2 \partial_1^3 \phi(0) - 2g \partial_1^2 \phi(0) + (1 + c) \partial_1 \phi(0) + g - c/g + c^3/g^2. \end{aligned} \quad (14)$$

The ‘initial’ data  $\partial_1^n \phi(0)$  are not totally independent, as can be seen from eq.(3). They are determined by requiring the appropriate analytic behavior of  $R(z)$ , which for small  $g$  must have a single cut on the real axis. This equation was derived in [11] (see also [10]).

In principle one can solve (13) iteratively for  $\phi$  in an expansion

$$\phi = \sum_{n=0} \phi^{(n)}(u_1, u_2), \quad (15)$$

where  $\phi^{(n)}$  contains all terms with  $n$  instances of  $u_2$ . Since there is only one term involving  $u_2$  explicitly, all terms containing  $\phi^{(n)}$  in the  $n$ -th recursion relation are linear in  $\phi^{(n)}$  and depend only on  $\phi_0$ . One can show that this differential operator, linear in  $\phi^{(n)}$ , has a unique inverse (under the condition that  $\phi^{(n)}$  is cyclic) and thus  $\phi^{(n)}$  is expressed in terms of  $\phi^{(m)}$  with  $m < n$ .

A simpler iterative scheme can be developed by starting with the (trivial) equation

$$\phi = 1 + u_1 \partial_1 \phi + u_2 \partial_2 \phi \quad (16)$$

and using (12) to eliminate the terms with  $\partial_2$  derivatives, producing

$$\phi = 1 + u_1 \partial_1 \phi + \frac{1}{c} u_2 (\partial_1 \phi - g \partial_1^2 \phi - \phi u_1 \phi). \quad (17)$$

Given  $\phi^{(n)}$  as in (15), this equation determines the terms in  $\phi^{(n+1)}$  with an initial  $u_2$  (by simply dropping the terms  $1 + u_1 \partial_1 \phi$ ). There is then a unique cyclically symmetric  $\phi^{(n+1)}$  which contains such terms.

It may be interesting to note that a purely algebraic equation can be derived for  $\phi$ . We start with (16) and its  $\partial_1$  derivatives

$$\begin{aligned} \phi &= 1 + u_1 \partial_1 \phi + u_2 \partial_2 \phi \\ \partial_1 \phi &= \partial_1 \phi(0) + u_1 \partial_1^2 \phi + u_2 \partial_2 \partial_1 \phi \\ \partial_1^2 \phi &= \partial_1^2 \phi(0) + u_1 \partial_1^3 \phi + u_2 \partial_2 \partial_1^2 \phi. \end{aligned} \quad (18)$$

Using (12) to eliminate the terms with  $\partial_2$  derivatives produces equations linear in  $\partial_1^2 \phi$ ,  $\partial_1^3 \phi$  and  $\partial_1^4 \phi$ . One then takes the system of these three equations and (13), and eliminates these higher derivatives between the equations, producing a single (non-linear) equation in terms of  $\phi$  and  $\partial_1 \phi$ . One then repeats the same derivation with the roles of  $u_1$  and  $u_2$  interchanged, producing an equation in terms of  $\phi$  and  $\partial_2 \phi$ . Finally, these two equations and (16) can be combined to eliminate  $\partial_1 \phi$  and  $\partial_2 \phi$ . We omit the final result due to its length.

These manipulations can be carried through despite the non-commuting nature of  $u_1$  and  $u_2$ . The final equation however contains the inverses  $u_1^{-1}$  and  $u_2^{-1}$  which require some discussion. First, the combinations  $u_1 u_2^{-1} u_1$  and  $u_2 u_1^{-1} u_2$  appear. This is incompatible with the expansion (15) and such terms must cancel in the final result. Second, the combination  $u_2^{-1} \phi$  appears. It is not clear to us that the iterative solution (15) is possible.

It would be interesting to express the results more explicitly. Of course, we considered (14) a complete solution of the truncated problem, even though its solution could be made

more explicit, because the techniques for solving such equations are so familiar. In practice one rarely even writes down its exact solution and instead studies limits of it, or its analytic behavior. Perhaps one should regard the solution of (13) in the same spirit.

We now turn to study a similar equation for the master field. One such equation [6] takes the form

$$\frac{\partial}{\partial M_i} S[\hat{M}] = \eta_i + [A, \hat{M}_i]. \quad (19)$$

To deal with this explicitly we must choose a representation for the master field. Many representations exist, in principle related by similarity transformations, and suited for different purposes. A very natural one is the ‘ $C$ -representation,’ [6,7] built from  $\psi(j)$ , the generating functional of connected planar correlation functions, a function of non-commuting variables  $j_i$ . [4] The master field  $\hat{M}_i$  is simply

$$\hat{M}_i = \frac{\partial}{\partial j_i} + \frac{\partial \psi(j)}{\partial j_i} \quad (20)$$

with the non-commuting derivative as above. If we define a trace as

$$\text{tr } \hat{O} = \hat{O} 1|_{j=0} = \langle |\hat{O}| \rangle \quad (21)$$

these operators are a master field.

The relation between the two generating functionals is [4]

$$\phi(u) = \psi(j), \quad j_i = u_i \psi(j). \quad (22)$$

and an equation with the same content as (19) was derived in [4] by simply changing variables from the Schwinger-Dyson equations. In general it is

$$\frac{\partial}{\partial M_i} S[\hat{M}]| \rangle = j_i | \rangle. \quad (23)$$

Comparing with (19) or the similar equation in [3,7], this equation is a bit simpler, as one bypasses the problem of finding the similarity transformation  $A$  (or representing the algebra  $[\hat{\Pi}_i, \hat{M}_j] = \delta_{ij} | \rangle \langle |$ ). On the other hand, those equations are representation-independent, while this one is tied to the  $C$ -representation. (The other difference between (19) and the other equations is that it produces cyclically symmetric equations which sum over all variations of  $M_i$  in the original path integral, while the others involve a single variation.)

For our case (23) becomes

$$\partial_1 \psi - c \partial_2 \psi - g \partial_1^2 \psi - g (\partial_1 \psi)^2 = j_1 \quad (24)$$

$$\partial_2 \psi - c \partial_1 \psi - g \partial_2^2 \psi - g (\partial_2 \psi)^2 = j_2. \quad (25)$$

We first derive an algebraic equation for  $\psi_0(j_1) = \psi(j_1, j_2 = 0)$  by a truncation analogous to (7). Truncating (24) gives

$$\begin{aligned} \partial_1 \psi_0 - c\psi_1 - g\partial_1(\psi_0 \partial_1 \psi_0) &= j_1 \\ \partial_1 \psi_1 - c\psi_2 - g\partial_1^2 \psi_1 - g\partial_1 \psi_1 \partial_1 \psi_0 - g\partial_1 \psi(0) \partial_1 \psi_1 &= 0 \end{aligned} \quad (26)$$

while (25) gives

$$\psi_1 - c\partial_1 \psi_0 - g\psi_2 - g(\psi_1)^2 = 0. \quad (27)$$

Again we have three equations for three unknowns, and it is easy to see that a quartic equation can be obtained for  $\psi_0$ .

In fact this equation can be directly related to (14). Let  $K(j_1) = 1/j_1 \psi_0(j_1)$  and use the reciprocity relations  $j_1 = R(z)$ ,  $z = K(j_1)$ . The algebraic equation  $F(z, R(z)) = 0$  is just  $F(K(j_1), j_1) = 0$ . From (14), it is a quartic equation in terms of  $K$ .

To solve (23) more generally, more steps are necessary. Applying  $\partial_2$  to (24), then replacing  $\partial_2^2 \psi$  by solving (25), we have

$$\begin{aligned} \partial_2 \partial_1 \psi - g\partial_2 \partial_1^2 \psi - g\partial_2 \partial_1 \psi \partial_1 \psi - g\partial_1 \psi(0) \partial_2 \partial_1 \psi \\ - \frac{c}{g} (\partial_2 \psi - c\partial_1 \psi - j_2) + c(\partial_2 \psi)^2 = 0. \end{aligned} \quad (28)$$

One can replace  $\partial_2 \psi$  in the above equation by solving the first equation of (24). To replace  $\partial_2 \partial_1 \psi$  and  $\partial_2 \partial_1^2 \psi$ , we apply  $\partial_1$  to the first equation from the right, and use the cyclic symmetry of coefficients of  $\psi$ . Thus,

$$\begin{aligned} \partial_2 \psi &= \frac{1}{c} (\partial_1 \psi - g\partial_1^2 \psi - g(\partial_1 \psi)^2 - j_1) \\ \partial_2 \partial_1 \psi &= \frac{1}{c} (\partial_1^2 \psi - g\partial_1^3 \psi - g\partial_1 \psi \partial_1^2 \psi - g\partial_1 \psi(0) \partial_1^2 \psi - 1) \\ \partial_2 \partial_1^2 \psi &= \frac{1}{c} (\partial_1^3 \psi - g\partial_1^4 \psi - g\partial_1 \psi \partial_1^3 \psi - g\partial_1^2 \psi(0) \partial_1^2 \psi - g\partial_1 \psi(0) \partial_1^3 \psi). \end{aligned} \quad (29)$$

Substituting these into (28), a single equation containing only derivatives with respect to  $j_1$  is obtained. This equation can be solved iteratively in the same way as (13). Using (20), this solution determines the master fields  $\hat{M}_i$  for this model. We intend to study these in more detail in subsequent work.

To summarize, we formulated and solved both large  $N$  Schwinger-Dyson equations and master field equations in the two matrix model, in terms of functions of non-commuting variables. The master field equation had the same basic structure as the Schwinger-Dyson equation and the same truncation method worked to solve it.

It is interesting to ask if truncation can work in more models. Another model which can be solved by the methods presented here has action  $S(M_1, M_2) = \text{tr} (M_1^2 + M_2^2 - cM_1M_2M_1M_2)/2$ , and we will discuss this and the general ‘meanders’ problem in future work.

There is no reason to think that similar truncations exist for all interesting models and it would be very desirable to develop more general techniques to work with this type of equation. For physics, it is often better to have qualitative and approximate approaches to understanding an equation, and if we can precisely define the equations, we can try to study these directions systematically. The results presented here should provide a useful test case.

It is also interesting to see whether one can go beyond the large  $N$  limit with the master field, for example to incorporate considerations in [13].

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